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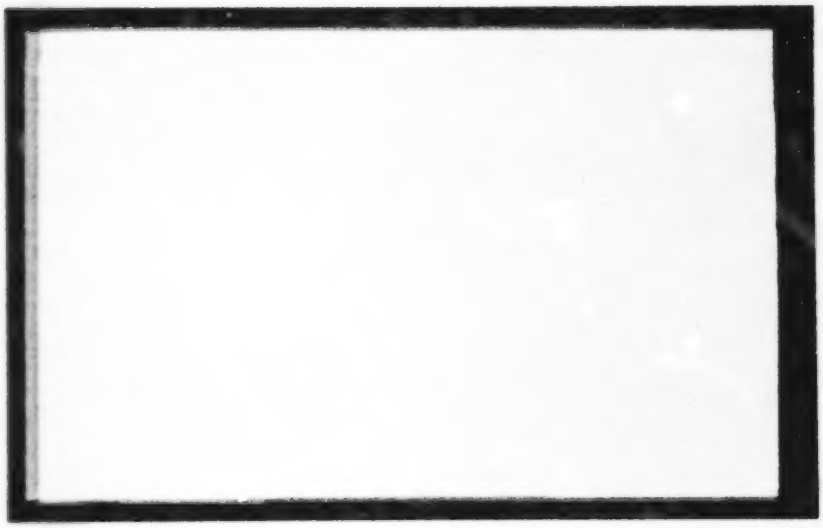

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THE GENERATION OF CHAIN BARRING CONSTRAINTS FOR MULTIDEPOT VEHICLE ROUTING PROBLEMS

by

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ABSTRACT

This paper provides an integer linear programming formulation for a class of multidepot vehicle routing problems. This formulation contains degree constraints, subtour elimination constraints, chain barring constraints and integrality constraints. The generation of chain barring constraints is described at length.

LA GENERATION DE CONTRAINTES D'ELIMINATION DE CHAÎNES DANS LES PROBLEMES DE TOURNEES COMPORTANT PLUSIEURS DEPÔTS

RESUME

Cet article décrit une formulation par programmation linéaire en nombres entiers d'une famille de problèmes de tournées comportant plusieurs dépôts. Cette formulation contient des contraintes de degré, des contraintes d'élimination de sous-tours, des contraintes d'élimination de chaînes et des contraintes d'intégralité. L'article décrit de façon exhaustive la génération des contraintes d'élimination de chaînes.

1. Introduction

The study of the travelling salesman problem (TSP) and of some of its extensions has led, in recent years, to major theoretical developments in the field of operational research and has provided new insights and solution techniques for a variety of location and vehicle routing problems encountered in real life situations. This paper examines an integer programming formulation for a class of TSP extensions called "multidepot vehicle routing problems".

We first introduce some notation and definitions. Let $G = (N,E)$ be a graph where $N = \{1,\ldots,n\}$ is a set of nodes (cities) and where E is a set of undirected edges. With E, we first associate a symmetrical cost or distance matrix $C^* = (c^*_{ij})$ and then $C = (c_{ij})$, the matrix of shortest distances between nodes of N. If C^* satisfies the triangle inequality, i.e. if $c^*_{ij} \leq c^*_{ik} + c^*_{kj}$ for all $i,j,k \in N$, then $C = C^*$. Note that C always satisfies the triangle inequality. Let $R \subseteq N$ be a set of *potential depots*. The number P of such nodes used as depots in the optimal solution must lie between two prespecified bounds $\underline{P} \geq 1$ and $\bar{P} \leq |R|$. The cost of using node r as a depot is equal to g_r. There are m_r identical vehicles based at depot r ($0 \leq \underline{m}_r \leq m_r \leq \bar{m}_r$), each with the same capacity D and a fixed cost f_r. To each node i of N-R, we associate a nonnegative *requirement* d_i ($\leq D$).

The problem consists of selecting depot sites (when $\underline{P} < |R|$), of determining how many vehicles are based at each selected depot and of establishing vehicle routes in such a way that

(i) each route starts and ends at the same depot;

(ii) all requirements are met exactly once by a vehicle (the same city i may be visited more than once if this saves distance but then, the requirement d_i of that city is satisfied only once by one vehicle);

(iii) the sum of all requirements satisfied by any vehicle does not exceed D;

(iv) $\underline{P} \leq P \leq \bar{P}$;

(v) for each node r used as a depot: $\underline{m}_r \leq m_r \leq \bar{m}_r$;

(vi) the total cost is minimized.

This problem is a generalization of many well known routing problems treated in the O.R. literature. In particular, it reduces to

(i) P1: the TSP [1] if $|R| = 1$, $\sum_{r \in R} m_r = 1$ and $\sum_{i \in N-R} d_i \leq D$;

(ii) P2: the m salesmen problem [2] if $|R| = 1$, $\sum_{r \in R} m_r = m \geq 1$ and $\sum_{i \in N-R} d_i \leq D$;

(iii) P3: the capacitated vehicle routing problem [3] if $|R| = 1$, $\sum_{r \in R} m_r = m \geq 1$ and $\sum_{i \in N-R} d_i > D$;

(iv) P4: the multidepot multisalesmen problem [4] if $\underline{P} = \bar{P} = |R| > 1$ and $\sum_{i \in N-R} d_i \leq D$;

(v) P5: the capacitated multidepot multisalesmen problem if $\underline{P} = \bar{P} = |R| > 1$ and $\sum_{i \in N-R} d_i > D$;

(vi) P6: a location-routing problem [5] if $\underline{P} < |R|$ and $\sum_{i \in N-R} d_i \leq D$;

(vii) P7: a capacitated location-routing problem if

$$\underline{P} < |R| \text{ and } \sum_{i \in N-R} d_i > D.$$

Our purpose is to provide a unified integer linear programming formulation for P7 (which represents the largest class of multidepot vehicle routing problems treated in this paper) and to analyze some aspects of this formulation.

2. Formulation

In addition to the notation already introduced, let us define

L : an arbitrarily large number

$$\lceil t \rceil = \begin{cases} \text{the smallest integer greater than or equal to } t & \text{if } t > 0 \\ 1 & \text{otherwise} \end{cases}$$

x_{ij} : a variable indicating the number of times edge (i,j) is used in the optimal solution. x_{ij} is not defined if $i \geq j$, if $i,j \in R$ or if $d_i + d_j > D$. In this paper, x_{ij} must be interpreted as x_{ji} whenever $i > j$.

y_r : a binary variable indicating whether node r is used as a depot ($y_r = 1$) or not ($y_r = 0$).

P7 can be formulated as follows:

(P7) minimize $z = \sum_{i,j \in N} c_{ij} x_{ij} + \sum_{r \in R} (g_r y_r + f_r m_r)$

subject to

(1) $\sum_{i<k} x_{ik} + \sum_{k<j} x_{kj} = 2$ $\quad (k \in N-R)$

(2) $\sum_{i<r} x_{ir} + \sum_{r<j} x_{rj} = 2m_r$ $\quad (r \in R)$

(3) $\sum_{i,j \in S} x_{ij} \leq |S| - \left\lceil \frac{\sum_{k \in S} d_k}{D} \right\rceil$ $\quad (S \subseteq N-R,\ |S| \geq 3)$

(4) $x_{i_1 i_2} + 3x_{i_2 i_3} + x_{i_3 i_4} \leq 4$ $\quad (i_1, i_4 \in R;\ i_2, i_3 \in N-R)$

(5) $x_{i_1 i_2} + x_{i_{h-1} i_h} + 2 \sum_{i,j \in R-\{i_1, i_h\}} x_{ij} \leq 2h-5$ $\quad (h \geq 5;\ i_1, i_h \in R;\ i_2, \ldots, i_{h-1} \in N-R;\ d_{i_2} + \cdots + d_{i_{h-1}} \leq D)$

(6) $y_r \leq m_r \leq L y_r$ $\quad (r \in R)$

(7) $\underline{m}_r \leq m_r \leq \bar{m}_r$ $\quad (r \in R)$

(8) $\underline{P} \leq \sum_{r \in R} y_r \leq \bar{P}$

(9) $y_r = 0, 1$ $(r \in R)$

(10) $x_{ij} = 0, 1$ $(i,j \in N\text{-}R)$

$x_{ij} = 0, 1, 2$ $(i \text{ or } j \in R)$

In this formulation, the m_r's can be considered as constants fixed a priori (in this case, the corresponding nodes r are necessarily chosen as depots whenever $m_r > 0$), or as variables if their value is determined simultaneously with the optimal depot sites and with the optimal routes. Constraints (1) and (2) specify the degree of each node. Constraints (1) state that each city of N-R is visited exactly once. This will in fact be the case if C* satisfies the triangle inequality; however, if it does not, the shortest path between i and j may well include other cities which will then be visited more than once, although this does not appear in the formulation. In this case, the requirement d_i of city i will be assigned only once to only one vehicle. Formulations for P1, P2 and P3 [1,2,3] are obtained by removing the y_r variables and constraints (4), (5), (6), (8), (9). Similarly, constraints (8) and (9) are unnecessary in problems P4 and P5 in which the set of depots is known in advance; in these two problems, constraints (6) reduce to

(6') $m_r \geq 1$ $(r \in R)$

Constraints (3) ensure that no solution contains subtours disconnected from R and no route having a total weight exceeding D. Their derivation is provided in [3] and in [6]. They reduce to the standard TSP subtour elimination constraints [1]:

(11) $\sum_{i,j \in S} x_{ij} \leq |S| - 1$ $(S \subseteq N\text{-}R,\ |S| \geq 3)$

in all problems for which $\sum_{i \in N-R} d_i \leq D$.

Our attention will be centered on multidepot problems which require an additional type of constraints, namely *chain barring constraints* (4) and (5). These constraints prohibit the formation of chains between any two depots and hence ensure that each route starts and ends at the same depot. Their derivation is described at length in the next section.

3. Chain barring constraints

Consider a relaxed problem P7' obtained from P7 by removing constraints (4) and (5). The optimal solution to P7' may contain *chains* between two depots, i.e. sequences of nodes $(i_1,\ldots,i_h)$ where

(i) $i_1, i_h \in R$; $i_2,\ldots,i_{h-1} \in N-R$;

(ii) $x_{i_t i_{t+1}} = 1 \qquad (t = 1,\ldots,h-1)$.

First assume that $h \geq 4$. (The two cases $h = 2$ and $h = 3$ are treated at the end of this section.) We wish to generate constraints that will eliminate such chains without cutting off the optimal solution to P7.

We describe a general method for deriving such constraints. Consider a chain $H = (i_1,i_2,\ldots,i_h)$. For $\ell > k$, we define $S_{k\ell} = \{i_k, i_{k+1},\ldots,i_\ell\}$. Let $X = x_{i_1 i_2}$, $Y = x_{i_{h-1} i_h}$ and $Z = \sum_{i,j \in S_{2,h-1}} x_{ij}$ and let $\bar{X}$, $\bar{Y}$, $\bar{Z}$ be the values taken by X, Y, Z respectively. In a feasible solution, $\bar{X}$, $\bar{Y} \in \{0, 1, 2\}$ and $\bar{Z}$ is an integer in $[0, u(\bar{X},\bar{Y})]$ where $u(\bar{X},\bar{Y})$ is the smallest upper bound on Z whose value depends on $\bar{X}$ and $\bar{Y}$. In the following development, it is assumed that the $u(\bar{X},\bar{Y})$'s are known constants. Later in this section, we indicate how these bounds are determined. Let

$$Q = \{(\bar{X},\bar{Y},\bar{Z}) : \bar{X},\bar{Y} \in \{0,1,2\},\ \bar{Z} \in [0,u(\bar{X},\bar{Y})],\ \bar{Z} \text{ integer}\}.$$

A valid cut is a constraint of the form

(12) $aX + bY + Z \leq d$

where $a,b \in \mathbb{R}$ and $d \geq 0$, which satisfies

(13) $d - a\bar{X} - b\bar{Y} \geq u(\bar{X},\bar{Y})$ $\qquad (\bar{X},\bar{Y} \in \{0,1,2\})$

while cutting the current solution characterized by $\bar{X} = 1$, $\bar{Y} = 1$ and $\bar{Z} = h-3$.

We seek the constraint which provides the deepest cut, i.e. the constraint that minimizes the value of $(d - a - b)$, the value taken by the left hand side of (13) when $\bar{X} = \bar{Y} = 1$. The coefficients of the cut are given by the solution of P*:

(P*) minimize $d - a - b$

subject to

(14) $d - a\bar{X} - b\bar{Y} \geq u(\bar{X},\bar{Y})$ $\qquad (\bar{X},\bar{Y} \in \{0,1,2\})$

(15) a, b free, $d \geq 0$

The optimal values of a, b and d can be obtained by introducing additional variables in P* or by solving the dual problem D*. Let the $v(\bar{X},\bar{Y})$'s represent the dual variables associated with constraints (14). The dual problem is

(D*) maximize $\sum_{\bar{X}=0}^{2} \sum_{\bar{Y}=0}^{2} u(\bar{X},\bar{Y})\, v(\bar{X},\bar{Y})$

subject to

$$(16)\quad \sum_{\bar{X}=0}^{2}\sum_{\bar{Y}=0}^{2} \bar{X}\, v(\bar{X},\bar{Y}) = 1$$

$$(17)\quad \sum_{\bar{X}=0}^{2}\sum_{\bar{Y}=0}^{2} \bar{Y}\, v(\bar{X},\bar{Y}) = 1$$

$$(18)\quad \sum_{\bar{X}=0}^{2}\sum_{\bar{Y}=0}^{2} v(\bar{X},\bar{Y}) \leq 1$$

$$(19)\quad v(\bar{X},\bar{Y}) \geq 0 \qquad (\bar{X},\bar{Y} \in \{0,1,2\})$$

The optimal values of a, b and d can also be determined otherwise, e.g. by using a physical model. When P* possesses an infinite number of optimal solutions, we only retain the constraints which define facets of Q. Note that the cut is only effective if it eliminates the current infeasible solution, i.e. if

$$(20)\quad d - a - b < h-3$$

As we shall see (relations (29)), this condition is always satisfied. We now proceed to the generation of chain barring constraints in some particular cases.

Chains involving 4 nodes

Consider a chain $H = (i_1, i_2, i_3, i_4)$. Here $X = x_{i_1 i_2}$, $Y = x_{i_3 i_4}$ and $Z = x_{i_2 i_3}$. We first observe that in any feasible solution to P7, Z can only take a positive value if $X + Y \leq 1$. Otherwise, edge (i_2, i_3) cannot enter the solution. The values of $u(\bar{X},\bar{Y})$ are therefore

$$(21)\quad u(\bar{X},\bar{Y}) = \begin{cases} 1 & \text{if } \bar{X} + \bar{Y} \le 1 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to verify that the optimal cutting plane (12) is uniquely determined by the following three points of Q: (0, 1, 1), (1, 0, 1) and (2, 2, 0). This plane can be expressed in terms of the x_{ij}'s as

$$(22)\quad x_{i_1 i_2} + 3x_{i_2 i_3} + x_{i_3 i_4} \le 4$$

Chains involving at least 5 nodes

Now consider a chain $H = (i_1, i_2, \ldots, i_h)$ where $h \ge 5$. As above, $X = x_{i_1 i_2}$, $Y = x_{i_{h-1} i_h}$ and $Z = \sum_{i,j \in S_{2,h-1}} x_{ij}$. Let us define $M_{k\ell}$ as the minimum number of vehicles required to visit all nodes of $S_{k\ell}$ in the optimal solution. It has been shown in [3] that $M_{k\ell}$ is bounded below by $\left\lceil \frac{\sum_{i \in S_{k\ell}} d_i}{D} \right\rceil$. Therefore, for any $S_{k\ell}$, the following inequality holds:

$$(23)\quad \sum_{i,j \in S_{k\ell}} x_{ij} \le |S_k| - M_{k\ell} = (\ell - k + 1) - M_{k\ell}$$

We will first illustrate the computation of $u(\bar{X},\bar{Y})$ by taking a special case. Let $\bar{X} = 2$ and $\bar{Y} \ne 2$; any feasible solution for P7 must satisfy

$$(24)\quad \sum_{j \in S_{3,h-1}} x_{i_2 j} = 0$$

and therefore

$$(25)\quad Z = \sum_{i,j \in S_{2,h-1}} x_{ij} = \sum_{i,j \in S_{3,h-1}} x_{ij} \le (h-3) - M_{3,h-1} = u(\bar{X},\bar{Y})$$

By applying the same kind of reasoning, it is possible to obtain bounds $u(\bar{X},\bar{Y})$ on Z for all combinations of $\bar{X}$ and $\bar{Y}$:

$$(26)\quad u(\bar{X},\bar{Y}) = \begin{cases} (h-2) - M_{2,h-1} & \text{if } \bar{X}+\bar{Y} \le 1 \text{ or } \bar{X}=\bar{Y}=1 \\ & \quad \text{and } M_{2,h-1} > 1 \\ (h-3) - M_{2,h-1} = h-4 & \text{if } \bar{X}=\bar{Y}=M_{2,h-1}=1 \\ (h-3) - M_{2,h-2} & \text{if } \bar{Y}=2 \text{ and } \bar{X}\neq 2 \\ (h-3) - M_{3,h-1} & \text{if } \bar{X}=2 \text{ and } \bar{Y}\neq 2 \\ (h-4) - M_{3,h-2} & \text{if } \bar{X}=\bar{Y}=2 \end{cases}$$

The case $u(\bar{X},\bar{Y}) = h-4$ if $\bar{X}=\bar{Y}=M_{2,h-1}=1$ requires some explanation: it is always valid to set $u(1,1) = (h-2) - M_{2,h-1}$; however, when $M_{2,h-1} = 1$, this becomes $u(1,1) = h-3$ which is equal to the current value of Z. In other words, by using this value for u(1,1), the plane defined by (12) and (13) does not cut the chain. It is therefore valid to set $u(1,1) = h-4$ in this case.

In order to find a chain barring constraint, it suffices

(i) to substitute in P* (or in D*) $\bar{X},\bar{Y}$ and $u(\bar{X},\bar{Y})$ by their values given by (26);

(ii) to express $M_{2,h-1}$, $M_{2,h-2}$ and $M_{3,h-1}$ in terms of $M_{3,h-2}$ by successively considering all seven possible cases:

a) $M_{3,h-2} = M_{3,h-1} = M_{2,h-2} = M_{2,h-1} = 1$

b) $M_{3,h-2} = M_{3,h-1} = M_{2,h-2} = M_{2,h-1} > 1$

c) $M_{2,h-2} = M_{3,h-1} = M_{3,h-2} + 1$; $M_{2,h-1} = M_{3,h-2} + 2$

d) $M_{2,h-2} = M_{3,h-1} = M_{2,h-1} = M_{3,h-2} + 1$

e) $M_{2,h-2} = M_{2,h-1} = M_{3,h-2} + 1;\quad M_{3,h-1} = M_{3,h-2}$

f) $M_{3,h-1} = M_{2,h-1} = M_{3,h-2} + 1;\quad M_{2,h-2} = M_{3,h-2}$

g) $M_{2,h-2} = M_{3,h-1} = M_{3,h-2};\quad M_{2,h-1} = M_{3,h-2} + 1;$

(iii) to make the change of variable $d' = d - (h - M_{3,h-2} - 4)$;

(iv) to solve P* (or D*).

The constraints resulting from this process are given in table 1.

A closer examination of table 1 reveals that in all cases but a), the first cutting plane can be expressed as

(27) $Z \leq h - 2 - M_{2,h-1}$

in other words,

(28) $$\sum_{i,j \in S_{2,h-1}} x_{ij} \leq |S_{2,h-1}| - \left\lceil \frac{\sum_{k \in S_{2,h-1}} d_k}{D} \right\rceil$$

which corresponds to constraint (3). We have just shown that in most cases, illegal chains can be broken by a generalized subtour elimination constraint; these constraints constitute facets of Q and provide the deepest cut (the minimal value of (d - a - b) is equal to $h - 2 - M_{2,h-1}$). Also note that in all cases, the cut is effective since

(29) $$d - a - b = \begin{cases} h - 3.5 < h - 3 & \text{in a)} \\ h - 2 - M_{2,h-1} < h - 3 & \text{in b) to g)} \end{cases}$$

TABLE 1

Chain barring constraints for all possible combinations of $M_{2,h-2}$, $M_{3,h-1}$ and $M_{2,h-1}$

Case	$M_{2,h-2}$	$M_{3,h-1}$	$M_{2,h-1}$	All optimal solutions of P* that constitute facets of Q
a)	1	1	1	$X + Y + 2Z \leq 2h - 5$
b)	$M_{3,h-2} > 1$	$M_{3,h-2} > 1$	$M_{3,h-2} > 1$	$Z \leq h - 2 - M_{3,h-2}$ $X + Z \leq h - 1 - M_{3,h-2}$ $Y + Z \leq h - 1 - M_{3,h-2}$ $X + Y + Z \leq h - M_{3,h-2}$
c)	$M_{3,h-2} + 1$	$M_{3,h-2} + 1$	$M_{3,h-2} + 2$	$Z \leq h - 4 - M_{3,h-2}$
d)	$M_{3,h-2} + 1$	$M_{3,h-2} + 1$	$M_{3,h-2} + 1$	$Z \leq h - 3 - M_{3,h-2}$ $X + Z \leq h - 2 - M_{3,h-2}$ $Y + Z \leq h - 2 - M_{3,h-2}$
e)	$M_{3,h-2} + 1$	$M_{3,h-2}$	$M_{3,h-2} + 1$	$Z \leq h - 3 - M_{3,h-2}$ $Y + Z \leq h - 2 - M_{3,h-2}$
f)	$M_{3,h-2}$	$M_{3,h-2} + 1$	$M_{3,h-2} + 1$	$Z \leq h - 3 - M_{3,h-2}$ $X + Z \leq h - 2 - M_{3,h-2}$
g)	$M_{3,h-2}$	$M_{3,h-2}$	$M_{3,h-2} + 1$	$Z \leq h - 3 - M_{3,h-2}$

Chains involving less than 4 nodes

The only chains left out of the above analysis are those involving 2 or 3 nodes. Illegal chains containing only 2 nodes cannot oc-

cur since x_{ij} is not defined when i and j $\in$ R. There is only one type of solution which could theoretically contain chains involving 3 nodes after all other types of chains have been eliminated: these would occur in subtours made up of an alternance of nodes from R and of nodes from N-R. Let $(i_1, j_1, i_2, j_2, \ldots, i_\ell, j_\ell, i_1)$ be such a subtour where $i_1, i_2, \ldots, i_\ell \in R$ and $j_1, j_2, \ldots, j_\ell \in$ N-R. However, we need not consider this case since it is never disadvantageous to replace this subtour by ℓ subtours of the form (i_k^*, i_k, i_k^*) where

$$(30)\quad i_k^* = \begin{cases} j_k & \text{if } c_{i_k j_k} \leq c_{i_k j_{k-1}} \qquad (j_0 = j_\ell) \\ j_{k-1} & \text{otherwise} \end{cases}$$

since

$$(31)\quad 2c_{i_k i_k^*} \leq c_{i_k j_{k-1}} + c_{i_k j_k}$$

4. Conclusion

We have provided a general integer programming formulation for a family of multidepot vehicle routing problems. All problems considered require degree constraints, subtour elimination constraints and integrality constraints. In addition, in problems involving more than one depot (P4 to P7) it may also be necessary to generate chain barring constraints. Our analysis shows that in all but two cases, the imposition of subtour elimination constraints suffices to eliminate illegal chains. Computational experiments on the use of our formulation to solve P7 will be presented in a forthcoming paper.

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POOR

QUALITY

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